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Abstract: For any graph G, the *k-improper chromatic number* $\chi^k(G)$ is the smallest number of colours used in a colouring of G such that each colour class induces a subgraph of maximum degree k. We investigate the ratio of the k-improper chromatic number to the clique number for unit disk graphs and random unit disk graphs to generalise results of [11, 8] and [10] where only proper colouring was considered.

Key-words: improper colouring, unit disk graph, random graph, chromatic number, radio channel assignment, clique number

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Coloration impropre des graphes (aléatoires) d'intersection de disques unitaires

Résumé : Pour tout graphe G, le *nombre chromatique k-impropre* est le plus petit nombre de couleurs nécessaires pour colorer les sommets de G de sorte que chaque classe de couleur induise un sous-graphe de degré maximum au plus k. Nous étudions le rapport entre le nombre chromatique k-impropre et la taille d'une plus grande clique pour les graphes d'intersection de disques unitaires et les graphes aléatoires d'intersection de disques unitaires. Nos résultats généralisent des résultats précédents concernant la coloration propre [11, 8, 10].

Mots-clés : coloration impropre, graphe d'intersection de disques, graphe aléatoire, nombre chromatique, assignation de fréquences, clique

1 Introduction

Given a set V of points in the plane and a distance threshold $r > 0$, we let $G(V, r)$ denote the following graph. The vertex set is V and distinct vertices are joined by an edge whenever the Euclidean distance between them is less than r. Any graph isomorphic to such a graph is called a *unit disk graph*. The study of the class of unit disk graphs stems partly from applications in communication networks. In particular, the problem of finding a proper vertex-colouring – in which the vertices of a graph are coloured so that adjacent vertices do not receive the same colour – of a given unit disk graph is closely associated with the so-called *frequency allocation problem* [2]. Consult [6] for a more general treatment of this important problem.

[11, 8, 10] and [12] investigated the chromatic number χ for unit disk graphs in two related cases. The first case is the asymptotic limit of χ where V is countably infinite and the distance threshold r approaches infinity: for countable sets V with finite upper density, the ratio of chromatic number over clique number approaches $2\sqrt{3}/\pi$ as $r \to \infty$ [11]. The second case is the asymptotic behaviour of χ for unit disk graphs based on randomly chosen points in the plane (where the distance threshold r approaches 0 as the number of points n approaches infinity). The papers [8, 10] establish almost sure (and in probability) convergence results for these random instances of unit disk graphs.

In this paper, we are also interested in vertex colourings of unit disk graphs; however, we partially relax the condition that any two vertices with the same colour may not be adjacent. Recall that, given an arbitrary colouring, a *colour class* is a set of vertices all assigned the same colour. Given $k \geq 0$, we say that a graph is *k-improper colourable* if there is a colouring in which each colour class induces a subgraph with maximum degree at most k. We wish to find the minimum number χ^k of colours used in such a colouring. Note that proper colouring is just 0-improper colouring and hence $\chi = \chi^0$. Our aim, in this paper, is to determine χ^k for unit disk graphs in the two cases mentioned above.

The problem of k-improperly colouring unit disk graphs arises in practice for instance when modelling certain satellite communications problems. More precisely, Alcatel Industries proposed the following problem: a satellite sends information to receivers on earth, each of which listens on a frequency. Technically, it is impossible to focus the signal sent by the satellite exactly on a receiver, so part of the signal is spread in an area around it creating noise for other receivers nearby and listening on the same frequency. A receiver is able to distinguish the signal directed to it from the noise it picks up if the sum of the noise does not exceed a certain threshold T. The problem is to assign frequency to the receivers in such a way that each receiver can obtain its dedicated signal properly. We consider the fundamental case where the noise areas at receivers are disks and where the "noise relation" is symmetric (that is, if a receiver u is in the noise area of a receiver v, then v is in the noise area of u). We assume that the radius r of a disk and the intensity I of the noise created by a signal are independent of the frequency and the receiver. Hence, to distinguish its signal from noises, a receiver must be in the noise area of at most $k = \lfloor T/I \rfloor$ receivers listening on the same frequency. We define a noise graph: the vertices are the receivers and we put an edge between u and v if u is in the noise area of v (and hence v in the noise area of u). Note that, according to the assumptions, the noise graph is a unit disk graph. The frequencies are represented by colours, so a valid assignment of frequencies to receivers is represented by a k-improper colouring and we aim to minimise the portion of radio spectrum used, i.e. find χ^k. The reader can refer to [4, 5] for further study of this problem.

Before going further, we must review and introduce some basic terminology and properties. We denote the maximum degree of G by $\Delta(G)$. A *clique* is a set of pairwise adjacent vertices; the *clique number* $\omega(G)$ is the maximum number of vertices in a clique of G. An *independent set* is a set of pairwise non-adjacent vertices; the *independence number* $\alpha(G)$ is the maximum number of vertices in an independent set of G. Recall that $|V(G)|/\alpha(G) \leq \chi(G)$. An analogous lower bound on $\chi^k(G)$ is as follows. A *k-dependent set* is a set of vertices whose induced subgraph has maximum degree at most k; the *k-dependence number* $\alpha^k(G)$ is the maximum number of vertices in a k-dependent set of G.

Proposition 1 *For any graph G and $k \geq 0$, $\chi^k(G) \geq |V(G)|/\alpha^k(G)$*

We leave the straightforward proof to the reader. Also recall that $\omega(G) \leq \chi(G) \leq \Delta(G)+1$ for any graph G. The following proposition shows that these bounds generalise to $\chi^k(G)$ in an appropriate sense (recalling that $\omega(G) \leq \chi(G)$):

Proposition 2 *For any graph G and $k \geq 0$, $\left\lceil \frac{\chi(G)}{k+1} \right\rceil \leq \chi^k(G) \leq \left\lceil \frac{\Delta(G)+1}{k+1} \right\rceil$.*

The second inequality is a corollary of a result due to Lovász [7]. To see that the first inequality holds, note that any colour class in a k-improper colouring induces a subgraph of maximum degree at most k and hence can be partitioned into $k+1$ independent sets.

Under the asymptotic models of unit disk graphs considered in this paper the lower bound in Proposition 2 more or less gives the right answer. We will see that in both models $(k+1)\chi^k$ approaches χ (in an appropriate sense, with some small exceptions). We mention here that under the standard asymptotic model for general graphs, i.e. the Erdős-Rényi $G(n,p)$ random graph model, there is qualitatively different behaviour. The perhaps rather counterintuitive result that, for $k = o(\ln n)$ and p fixed, $\chi^k(G(n,p))/\chi(G(n,p)) \to 1$ almost surely as $n \to \infty$ has been proved [9] (recall that if $Z, Z_1, Z_2, \ldots$ are random variables then we say that Z_n tends to Z almost surely if $\mathbb{P}(Z_n \to Z) = 1$).

We note that the clique number of a unit disk graph can be found in polynomial time by means of an $\mathrm{O}(n^{4.5})$ algorithm [1] and even when an explicit representation in the plane is not available [16]. In contrast, the problem of finding the chromatic number of unit disk graphs is NP-complete [1]. Recent work [3] shows that the same holds for the k-improper chromatic number χ^k, for any fixed k. Furthermore, by the above two propositions and since $\Delta(G) \leq 6\omega - 6$ for any unit disk graph G (to see this, consider that each $\pi/3$-sector of a unit disk induces a clique), we have a heuristic for χ^k with approximation ratio 6. Except for the case $k = 0$ (where [13] gives a 3-approximation), 6 is the best known approximation ratio for χ^k.

The paper is divided as follows. In Sections 2 and 3, we consider the extensions of [11], stating definitions and results, then later giving some indication to the proofs. Similarly, in Sections 4 and 5, we analyse improper colouring for random geometric graphs to extend results of [8] and [10].

2 Asymptotically, improperly colouring unit disk graphs

This section discusses our extensions of [11]. Let V be any countable set of points in the plane. For $x > 0$, let $f(x)$ be the supremum of the ratio $|V \cap S|/x^2$ over all open $(x \times x)$ squares S with sides aligned with the axes. The *upper density* of V is $\sigma^+(V) = \inf_{x>0} f(x)$.

Theorem 1 ([11]) *Let V be a countable non-empty set of points in the plane with upper density $\sigma^+(V) = \sigma$. Then $\omega(G(V,r))/r^2 \geq \sigma\pi/4$ and $\chi(G(V,r))/r^2 \geq \sigma\sqrt{3}/2$ for any $r > 0$; and, as $r \to \infty$, we have $\Delta(G(V,r))/r^2 \to \sigma\pi$, $\omega(G(V,r))/r^2 \to \sigma\pi/4$ and $\chi(G(V,r))/r^2 \to \sigma\sqrt{3}/2$.*

Note that $2\sqrt{3}/\pi \approx 1.103$. We extend this theorem as follows.

Theorem 2 *Let V be a countable non-empty set of points in the plane with upper density $\sigma^+(V) = \sigma$. Then $\chi^k(G(V,r))/r^2 \geq \frac{\sigma\sqrt{3}/2}{k+1}$ for any $r > 0$ and, as $r \to \infty$, we have $(k+1)\chi^k(G(V,r))/r^2 \to \sigma\sqrt{3}/2$ if $k = o(r)$.*

In particular the following holds:

Corollary 1 *Let $V \subseteq \mathbb{R}^2$ be a set with upper density $0 < \sigma < \infty$ and suppose k satisfies $k = o(r)$. Then*

$$\frac{(k+1)\chi^k(G(V,r))}{\chi(G(V,r))} \to 1,$$

as $r \to \infty$.

It also holds that, for any countable set V of points in the plane with finite positive upper density, the ratio of $\chi^k(G(V,r))$ to $\omega(G(V,r))/(k+1)$ tends to $2\sqrt{3}/\pi$ as r approaches infinity. We have allowed k to vary as a function of r, but this does not detriment our results.

McDiarmid and Reed also tighten the upper bounds in Theorem 1 for the case where the points are approximately uniformly spread over the plane. Given a set V of points in the plane, a *cell structure* of V with density σ and radius ρ is a family $(C_v : v \in V)$ of sets that partition the plane and such that each C_v has area $1/\sigma$ and is contained in a ball of radius ρ about v.

Theorem 3 ([11]) *Let the set V of points in the plane have a cell structure with density σ and radius ρ. Then, for any $r > 0$, we have $\omega(G(V,r)) \leq (\sigma\pi/4)(r+2\rho)^2$ and $\chi(G(V,r)) < ((\sigma\sqrt{3}/2)^{1/2}(r+2\rho) + (2/\sqrt{3}) + 1)^2$. Thus, (combined with Theorem 1), $\omega(G(V,r)) = (\sigma\pi/4)r^2 + \mathrm{O}(r)$ and $\chi(G(V,r)) = (\sigma\sqrt{3}/2)r^2 + \mathrm{O}(r)$ as $r \to \infty$.*

We extend this theorem as follows.

Theorem 4 *Let the set V of points in the plane have a cell structure with density σ and radius ρ. Then, for any $r > 0$,*

$$\begin{aligned}\chi^k(G(V,r)) \quad < \quad & ((\sigma\sqrt{3}/2)^{1/2}(r+2\rho) + (2/\sqrt{3}) + 1) \\ & \quad ((\sigma\sqrt{3}/2)^{1/2}(r+2\rho) + (2/\sqrt{3}) + 2k + 1)/(k+1).\end{aligned}$$

Thus, $(k+1)\chi^k(G(V,r)) = (\sigma\sqrt{3}/2)r^2 + \mathrm{O}(kr)$ as $r \to \infty$.

The key to all of the above theorems is the special case when V is the triangular lattice T, which is defined as the integer linear combinations of the vectors $a = (1,0)$ and $b = (1/2, \sqrt{3}/2)$. Let G_T denote the graph whose vertex set is T and two vertices adjacent whenever they are at distance 1 from each other. Note that the Dirichlet-Voronoï cells of the set T constitute a cell structure with density $2/\sqrt{3}$ and radius $1/\sqrt{3}$, and hence Theorem 4 above gives good bounds on $\chi^k(G(V,r))$. However, we can obtain better results, and, indeed, for $k = 0$, there is an exact result.

For any $r > 0$, let $\hat{r}$ be the minimum distance between two points in T subject to that distance being at least r (i.e. $\hat{r}$ is the least value of $(x^2 + xy + y^2)^{1/2}$ greater than or equal to r so that x, y are non-negative integers). Note that $r \leq \hat{r} \leq \lceil r \rceil$, and the value of $\hat{r}^2$ can be computed in $\mathrm{O}(r)$ arithmetic operations.

Theorem 5 ([11]) *For any $r > 0$, $\chi(G(T,r)) = \hat{r}^2$.*

Consult [11] for the origin of this result. Unfortunately, when we consider k-improper colouring, we do not obtain an exact result such as Theorem 5, but we have a good bound:

Theorem 6 *For any $r > 0$ and $k \geq 0$,*

$$\chi^k(G(T,r)) \leq \lceil r \rceil \left(\left\lceil \frac{\lceil r \rceil - 1}{k+1} \right\rceil + 1 \right) < \frac{(r+1)(r+2k+1)}{k+1};$$

furthermore, if $r < \lceil (k+1)/2 \rceil$, then $\chi^k(G(T,r)) \leq \lceil 2r/\sqrt{3} \rceil$.

3 Proofs for Section 2

As mentioned in Section 2, the main results rest on the special case when V is the set of points on the triangular lattice T so we will first focus our attention here:

Proof of Theorem 6. We just need to exhibit a k-improper colouring of T that satisfies the bound. It turns out that a *strict tiling* of T – a colouring such that each colour class is a translate $v + T'$ of some sublattice T' of T – suffices. We can describe such a colouring succinctly by using one of its "tiles" – a finite subset $V' \subseteq T$ such that $V' + T'$ both covers and packs T.

Let us first define the tile V' and the sublattice T'. Set

$$x_0 = (k+1)\left(\left\lceil \frac{\lceil r\rceil - 1}{k+1} \right\rceil + 1\right) \text{ and } y_0 = \lceil r\rceil.$$

We define V' to be all points $xa + yb$ such that $(0,0) \leq (x,y) < (x_0, y_0)$ and T' to be all integer linear combinations of $x_0 a$ and $y_0 b$. Clearly, $V' + T'$ both covers and packs T.

Define $V'_{i,j} = \{(i',j) \mid i(k+1) \leq i' \leq (i+1)(k+1)-1\}$ for $(0,0) \leq (i,j) < (x_0/(k+1), y_0)$ and assign each of the $V'_{i,j}$s a distinct colour. By the choice of x_0 and y_0, it is simple to see that this extends to a k-improper colouring of T and a simple calculation shows that the number of colours is as given.

The "furthermore" condition implies that a colouring with one colour per row of T suffices and hence we need at most $\lceil 2r/\sqrt{3}\rceil$ colours. □

Now, for the lower bound of Theorem 2 (and hence of Theorem 4), it is possible to mimic the approach given in [11] by establishing a lower bound on a k-improper analogue of the *stability quotient* – the maximum over all induced subgraphs $H \subseteq G$ of $|V(H)|/\alpha(H)$ – however, it suffices to apply Proposition 2 to Theorem 1. We just need to prove upper bounds. It is possible to generalise the arguments of [11].

Let us recall a definition from [11]. Given two sets A and B of points in the plane, and $w > 0$, we say that a function $\phi : A \to B$ is *w-wobbling* if the Euclidean distance $d(a, \phi(a))$ is at most w for each $a \in A$. Observe that, if there is a w-wobbling injection from A into B, then $\chi^k(G(A,r)) \leq \chi^k(G(B, r+2w))$ for any $r > 0$.

Proof of Theorem 2. We will simply apply the proof of Lemma 11 in [11]. Let $\varepsilon > 0$. We wish to show that $\chi^k(G)/r^2 \leq (\sigma+\varepsilon)\sqrt{3}/\{2(k+1)\}$. First, we set T' to be T scaled so that its density is $(\sigma + \varepsilon/2)$, i.e. let $T' = \gamma^{-1}T$ where $\gamma = ((\sigma+\varepsilon/2)\sqrt{3}/2)^{1/2}$.

Let S denote the half-open unit square $S = \{(x,y) \mid 0 \leq x, y < 1\}$. For any x sufficiently large, every translate of the square xS contains at least $(\sigma+\varepsilon/4)x^2$ points of T' and at most this number of points of V. If we partition the plane into a square grid with side length x, then for each grid square X there is a w-wobbling injection from $V \cap X$ into $T' \cap X$ where $w = \sqrt{2}x$. We may patch these injections together to obtain a w-wobbling injection $\phi : V \to T'$.

Now, using Theorem 6, we obtain

$$\begin{aligned}
\chi^k(G(V,r)) &\leq \chi^k(G(T', r+2w)) \\
&= \chi^k(G(T, \gamma(r+2w))) \\
&< (\gamma(r+2w)+1)(\gamma(r+2w)+2k+1)/(k+1) \\
&< (\sigma+\varepsilon)\sqrt{3}/\{2(k+1)\}
\end{aligned}$$

if r is sufficiently large. □

Proof of Theorem 4. We will apply the proof of (3) in Theorem 2 in [11]. We first recall that the cells of the triangular lattice T constitute a cell structure with density $2/\sqrt{3}$ and radius $1/\sqrt{3}$. Now $\left(\frac{3}{4}\right)^{1/4}\sigma^{1/2}V$ has the same density $2/\sqrt{3}$, but has radius $\left(\frac{3}{4}\right)^{1/4}\sigma^{1/2}\rho$. By Lemma 13 of [11], $\left(\frac{3}{4}\right)^{1/4}\sigma^{1/2}V$ and T are *w-close* – there is a w-wobbling bijection between them – where $w = 1/\sqrt{3} + \left(\frac{3}{4}\right)^{1/4}\sigma^{1/2}\rho$ and hence, for any $r > 0$,

$$\chi^k(G(V,r)) = \chi^k\left(G\left(\left(\frac{3}{4}\right)^{1/4}\sigma^{1/2}V, \left(\frac{3}{4}\right)^{1/4}\sigma^{1/2}\rho\right)\right) \le \chi^k(G(T,D))$$

where $D = \left(\frac{3}{4}\right)^{1/4}\sigma^{1/2}r + 2w = \left(\frac{3}{4}\right)^{1/4}\sigma^{1/2}(r+2\rho) + \frac{2}{\sqrt{3}}$, whence,

$$\chi^k(G(V,r)) \le \chi^k(G(T,D)) < \frac{(D+1)(D+2k+1)}{k+1}.$$

by Theorem 6. □

4 Improper colouring of random unit disk graphs

This section discusses our generalisations of [8] and [10]. We consider graphs G_n obtained as follows. We pick vertices $X_1, \ldots, X_n \in \mathbb{R}^2$ at random (i.i.d. according to some probability distribution ν on $\mathbb{R}^2$) and we set $G_n = G(\{X_1, \ldots, X_n\}, r(n))$, where we assume we are given a sequence of distances $r(n)$ that satisfies $r(n) \to 0$ as $n \to \infty$. We will allow any choice of ν that has a bounded density. We are interested in the behaviour of the clique number, the chromatic number, and the k-improper chromatic number of G_n as n grows large.

In this model, the distance $r(n)$ plays a role similar to that of $p(n)$ in the Erdős-Rényi $G(n,p)$ model. Depending on the choice of $r(n)$, qualitatively different types of behaviour can be observed. We prefer to describe the various cases in terms of the quantity r^2n, because r^2n can be considered a measure of the average degree of the graph. Intuitively, this should be obvious. (Consider for instance the case ν is uniform on $[0,1]^2$, so that the probability of an edge between X_1 and X_2 is $\approx \pi r^2$ when r is small and the expected degree of X_1 is therefore $\approx \pi(n-1)r^2$.) For a somewhat more rigorous treatment of the relationship between nr^2 and the average degree, see [10].

In this section we will we only consider the case when the parameter k is fixed. It is however possible to generalise the proofs to growing k as long as k does not grow too quickly. The results fully extend to arbitrary norm and dimension, i.e. the case when points are drawn from some distribution on $\mathbb{R}^d$ (just replace 2 by d in what follows) and an arbitrary norm is used to measure the distance between points. However, the scope of this paper is unit disk graphs on the plane.

The following result was alluded to in the introduction. Let us say that the sequence $r(n)$ with $r \to 0$ satisfies the condition $(*)$ if either $nr^2 \gg \ln(n)$ or $nr^2 \sim t\ln(n)$ for some $t \in (0,\infty)$ or $n^{-\epsilon} \ll nr^2 \ll \ln(n)$ for all $\epsilon > 0$.

Theorem 7 *For $k \geq 0$ fixed and G_n as before the following holds:*

(i) *If r satisfies the condition $(*)$ then*

$$\frac{(k+1)\chi^k(G_n)}{\chi(G_n)} \to 1 \text{ almost surely;}$$

(ii) *If $nr^2 \ll n^{-\epsilon}$ for some $\epsilon > 0$ then*

$$\mathbb{P}\left(\chi^k(G_n) \in \left\{\left\lceil \frac{\chi(G_n)}{k+1} \right\rceil, \left\lceil \frac{\chi(G_n)}{k+1} \right\rceil + 1\right\} \text{ for all but finitely many } n\right) = 1.$$

This following proposition shows that the two point range for $\chi^k(G_n)$ in item **(ii)** cannot be reduced in general:

Proposition 3 *If $k \geq 1$ is fixed and r is chosen so that $nr^2 = \gamma n^{-\frac{1}{m(k+1)}}$ for some $\gamma > 0, m \in \mathbb{N}^*$, then there exists a $c = c(\gamma, m) \in (0,1)$ such that*

$$\mathbb{P}\left(\chi^k(G_n) = \left\lceil \frac{\chi(G_n)}{k+1} \right\rceil + 1\right) \to c.$$

When $nr^2 \ll n^{-\epsilon}$ for some $\epsilon > 0$ then it can be shown that $\chi^k(G_n)$ will remain bounded in the sense that $\mathbb{P}(\chi^k(G_n) \leq m$ for all but finitely many $n) = 1$ for some $m = m(\epsilon)$. Thus, proposition 3 shows that when $nr^2 \ll n^{-\epsilon}$, almost sure convergence of the ratio $(k+1)\chi^k(G_n)/\chi(G_n)$ to 1 does not hold in general.

In contrast it was shown in [10] that for proper colouring it holds that $\mathbb{P}(\chi(G_n) = \omega(G_n)$ for all but finitely many $n) = 1$ whenever $nr^2 \ll n^{-\epsilon}$ for some $\epsilon > 0$.

It follows from the proof of Theorem 7 that when $nr^2 \ll n^{-\epsilon}$ for some $\epsilon > 0$ then there exists a sequence $m(n)$ such that $\mathbb{P}(\chi^k(G_n) \in \{m(n), m(n)+1\}$ for all but finitely many $n) = 1$. Thus the probability distribution of χ^k becomes concentrated on two consecutive integers as n grows large in the sense that $\mathbb{P}(\chi^k(G_n) \in \{m(n), m(n)+1\}) \to 1$.

This phenomenon (of the probability measure becoming concentrated on two consecutive integers) is dubbed focusing in [14, 15] and is well known to occur for various graph parameters in Erdős-Rényi random graphs. Recently, one of the authors proved a conjecture of Penrose stating that when $nr^2 \ll \ln n$ then the clique number becomes focused and the same was shown to hold for the chromatic number ([12]). The proof can be easily adapted to also yield the analogous result for improper colouring: if $nr^2 \ll \ln(n)$ then there exists a sequence $m(n)$ such that $\mathbb{P}(\chi^k(G_n) \in \{m(n), m(n)+1\}) \to 1$.

5 Proofs for Section 4

The proof of item **(ii)** of Theorem 7 makes use of the following results from [10]:

Theorem 8 ([10]) *If $nr^2 \ll n^{-\epsilon}$ for some $\epsilon > 0$ then the following hold:*

(i) *There exists a sequence $m(n)$ such that $\mathbb{P}(\omega(G_n) \in \{m(n), m(n)+1\}$ and $\Delta(G_n) \in \{m(n), m(n)-1\}$ for all but finitely many $n) = 1$;*

(ii) $\mathbb{P}(\chi(G_n) = \omega(G_n)$ *for all but finitely many* $n) = 1$.

Theorem 7, item **(ii)** is an easy consequence of this last theorem:

Proof of Theorem 7, item (ii). It follows from item **(i)** that when $nr^2 \ll n^{-\epsilon}$ then

$$\mathbb{P}(\Delta(G_n) \in \{\omega(G_n), \omega(G_n)-1\} \text{ for all but finitely many } n) = 1.$$

Combining this with part **(ii)** of Theorem 8, we see that also

$$\mathbb{P}(\chi(G_n) = \omega(G_n) \text{ and } \Delta(G_n) \le \omega(G_n) \text{ for all but finitely many } n) = 1.$$

Now note that if $\chi(G_n) = \omega(G_n)$ and $\Delta(G_n) \le \omega(G_n)$ then Proposition 2 gives

$$\left\lceil \frac{\chi(G_n)}{k+1} \right\rceil \le \chi^k(G_n) \le \left\lceil \frac{\chi(G_n)+1}{k+1} \right\rceil \le \left\lceil \frac{\chi(G_n)}{k+1} \right\rceil + 1,$$

which concludes the proof. □

For the proof of Proposition 3 we will rely on some results from Chapter 3 of [15]. Recall that if Z, Z' are two integer valued random variables then their *total variational distance* is defined as

$$d_{TV}(Z, Z') = \sup_{A \subseteq \mathbb{Z}} |\mathbb{P}(Z \in A) - \mathbb{P}(Z' \in A)|.$$

Proposition 4 ([15]) *Let H be a connected unit disk graph with $l \ge 2$ vertices, let N denote the number of induced subgraphs of G_n isomorphic to H and let Z be a poisson variable with mean $\mathbb{E}N$. The following hold:*

(i) *There exists a constant $\mu = \mu(H) > 0$ such that $\mathbb{E}N \sim \mu n^l r^{2(l-1)}$;*

(ii) *There exists a constant $c = c(H)$ such that $d_{TV}(N, Z) \le cnr^2$.*

Proposition 5 ([15]) *Let $H_1, \ldots, H_s$ be non-isomorphic connected unit disk graphs with $l \ge 2$ vertices. Let N_i denote the number of induced subgraphs of G_n isomorphic to H_i. Let $\mu_1, \ldots, \mu_m$ be as given by part* **(i)** *of Proposition 4. Suppose that $nr^2 \sim \gamma n^{-\frac{1}{l-1}}$ with $\gamma > 0$. Let $Z_1, \ldots, Z_s$ be independent poisson variables with $\mathbb{E}Z_i = \gamma^{l-1}\mu_i$. Then*

$$(N_1, \ldots, N_s) \xrightarrow{d} (Z_1, \ldots, Z_s).$$

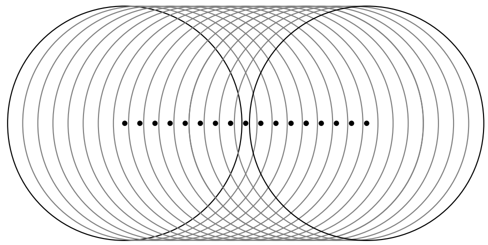

Figure 1: For Proposition 3, $H = K_{m(k+1)+1}\backslash e$ satisfies $\omega(H) = m(k+1)$ and $\chi^k(H) = m+1$.

Proof of Proposition 3. By part **(ii)** of Theorem 8 it suffices to consider $\mathbb{P}(\chi^k(G_n) = \left\lceil \frac{\omega(G_n)}{k+1} \right\rceil + 1)$, as $\mathbb{P}(\chi(G_n) \neq \omega(G_n)) \to 0$. Set $l = m(k+1)+1$. By choice of $r(n)$ we have that $n^l r^{2(l-1)} = \gamma^{l-1}$, and that $n^{l+1}r^{2l} \to 0, n^{l-1}r^{2(l-2)} \to \infty$. If we denote the order of the largest component of G_n by $L(G_n)$ then Proposition 4 implies that

$$\mathbb{P}(\omega(G_n) \geq l-1, L(G_n) \leq l) \to 1.$$

To see this let N be the number of induced subgraphs of G_n isomorphic to K_{l-1} and let N' be the number of connected subgraphs of order $l+1$. Part **(i)** of Proposition 4 gives $\mathbb{E}N' = O(n^{l+1}r^{2l}) = o(1)$, so that

$$\mathbb{P}(L(G_n) > l) = \mathbb{P}(N' > 0) \leq \mathbb{E}N' = o(1).$$

On the other hand $\mathbb{E}N = \Omega(n^{l-1}r^{2(l-2)}) \to \infty$, and using part **(ii)** of Proposition 4 we get:

$$\mathbb{P}(\omega(G_n) < l-1) = \mathbb{P}(N=0) = \exp[-\mathbb{E}N] + o(1) = o(1).$$

Let $H_1, \ldots, H_s$ be all non-isomorphic connected unit disk graphs of order l that satisfy $\chi^k(H_i) = m+1$ yet H_i is not (isomorphic to) K_l. There exists at least one such graph, the unit disk graph $H := G(\{(\frac{i}{l-1}, 0) : i = 0, \ldots, l-1\}, 1)$ (depicted in figure 5). This is simply the complete graph on l vertices with one edge removed. To see that there is no k-improper colouring of H with m colours, note that its vertices can be partitioned into a clique of size $l-1 = m(k+1)$ and a vertex v_0 which is adjacent to all but one of the other nodes. If there were a k-improper colouring with m colours then every colour would have to occur $k+1 \geq 2$ times amongst the vertices of the clique. Hence whichever of the m colours we assign to v_0 there will be a node in the clique adjacent to $k+1$ nodes of the same colour.

Let N_0 be the number of induced subgraphs of G_n isomorphic to K_l and let N_i be the number of induced subgraphs isomorphic to H_i. Observe that if both $\omega(G_n) \geq l-1$ and $L(G_n) \leq l$ hold then $\chi^k(G_n) = \left\lceil \frac{\chi(G_n)}{k+1} \right\rceil + 1$ holds if and only if $N_0 = 0$ and $N_i > 0$ for some $1 \leq i \leq s$. This gives

$$\mathbb{P}(\chi^k(G_n) = \left\lceil \frac{\chi(G_n)}{k+1} \right\rceil + 1) = \mathbb{P}(N_0 = 0 \text{ and } N_i > 0 \text{ for some } 1 \leq i \leq s) + o(1).$$

Using Proposition 5 we may therefore conclude that

$$\mathbb{P}(\chi^k(G_n) = \left\lceil \frac{\chi(G_n)}{k+1} \right\rceil + 1) \to e^{-\mu_0 \gamma^{l-1}}(1 - e^{-(\mu_1 + \cdots + \mu_s)\gamma^{l-1}}),$$

for some $\mu_0, \ldots, \mu_s > 0$. □

The proof of item **(i)** of Theorem 7 relies on some results from [10] that were developed to study the behaviour of $\chi(G_n)$.

One important ingredient to the proof is the connection between graph colouring and integer linear programming. Recall that the chromatic number of a graph G is the optimum value of the following integer linear program (ILP for short):

$$\begin{array}{rl} \min & 1^T x \\ \\ \text{subject to} & Ax \geq 1, \\ & x \geq 0, x \text{ integers,} \end{array}$$

where A is the *vertex-independent set incidence matrix* of G. This is a $(0,1)$-matrix whose rows are indexed by the vertices of G and whose columns correspond to all possible independent sets in G. It has $a_{ij} = 1$ if vertex v_i is in the independent set corresponding to the j-th column and $a_{ij} = 0$ otherwise. Now, given a nonnegative integer vector $b = (b_1, \ldots, b_n)$, let the graph G' be obtained from G by replacing vertex $v_i \in G$ by a clique of size b_i and the vertices in the cliques corresponding to v_i and v_j are joined in G' if and only if v_i and v_j are joined in G. Then $\chi(G')$ is the objective value of the following ILP:

$$\begin{array}{rl} \min & 1^T x \\ \\ \text{subject to} & Ax \geq b, \\ & x \geq 0, x \text{ integers} \end{array}$$

Furthermore $\chi^k(G')$ does not exceed the objective value of the ILP:

$$\begin{array}{rl} \min & 1^T x \\ \\ \text{subject to} & (k+1)Ax \geq b, \\ & x \geq 0, x \text{ integers} \end{array}$$

This is because taking $k+1$ copies of a each node in a stable set in G gives a k-dependent set in G' (but not every k-dependent set can be constructed in this way of course).

As mentioned in Section 4 we assume that the probability measure ν on the plane used to generate the X_i has a bounded density function f. Let us denote the *essential supremum* of f by $f_{\max}$, i.e.:

$$f_{\max} := \sup\{t : |\{x : f(x) > t\}| > 0\},$$

where $|.|$ denotes the Lebesgue measure. We say that a measurable set $A \subseteq \mathbb{R}^2$ has a *small neighbourhood* if $\lim_{\epsilon \searrow 0} |A_\epsilon| = |A|$ where A_ϵ denotes $A + B(0;\epsilon)$.

For φ a bounded, nonnegative function with essential supremum $0 < \varphi_{\max} < \infty$ and bounded support that satisfies the regularity conditon that $\{x : \varphi(x) > t\}$ has a small neighbourhood for all t, let us define the random variable M_φ as:

$$M_\varphi := \max_{x \in \mathbb{R}^2} \sum_{i=1}^{n} \varphi\left(\frac{X_i - x}{r}\right).$$

It turns out that the random variables M_φ play an important role when studying the (k-improper) chromatic number of G_n, see [10].

Proposition 6 ([10]) *If φ is as above then the following hold:*

(i) *If $nr^2 \ll \ln(n)$ yet $nr^2 \gg n^{-\epsilon}$ for all $\epsilon > 0$ then*

$$\frac{M_\varphi}{\ln(n)/\ln\left(\frac{\ln(n)}{nr^2}\right)} \to \varphi_{\max} \text{ almost surely;}$$

(ii) *If $nr^2 \gg \ln(n)$ then*

$$\frac{M_\varphi}{nr^2} \to f_{\max} \int_{\mathbb{R}^2} \varphi(x)dx \text{ almost surely;}$$

(iii) *If $nr^2 \sim t\ln(n)$ for some $t \in (0,\infty)$ then*

$$\frac{M_\varphi}{nr^2} \to f_{\max} \int_{\mathbb{R}^2} \varphi(x)e^{\varphi(x)s}dx \text{ almost surely,}$$

where $s = s(\varphi, t) \geq 0$ solves

$$\int_{\mathbb{R}^2} (s\varphi(x)e^{\varphi(x)s} - e^{\varphi(x)s} + 1)dx = \frac{1}{tf_{\max}}.$$

Proof of Theorem 7, item (i). We will first derive an upper bound on $\chi^k(G_n)$. To this end, let us consider the graphs G'_n constucted as follows. Dissect $\mathbb{R}^2$ into squares of side ϵr. Let Γ be the graph with vertex set $\epsilon r\mathbb{Z}^2$ and an edge $p \sim q$ if $\| p - q \| < r(1 + \epsilon\sqrt{2})$. For

$p \in \mathbb{R}^2$ let us denote by $N(p)$ the number of points in the square with lower left-hand corner p, i.e.:

$$N(p) := |\{X_1, \ldots, X_n\} \cap (p + [0, \epsilon r)^2)|.$$

We will now consider the graph G'_n, constructed by replacing each point p of Γ by a clique of size $N(p)$. Note that G_n is a subgraph of G'_n, so in particular $\chi^k(G_n) \leq \chi^k(G'_n)$. Let us fix $K > 0$ (large) such that ϵ divides K. For $p \in \epsilon r\mathbb{Z}^2$, let us denote by H_p the subgraph of G_n induced by the points in $p + [0, Kr)^2$ and let H'_p denote the corresponding subgraph of G'_n. Let A be the vertex-independent set incidence matrix of the subgraph Γ_K of Γ induced by the points of Γ inside $[0, Kr)^2$. By previous observations we know that $\chi^k(H'_p)$ is no more than the value of the ILP

$$\begin{array}{ll} \min & 1^T x \\ & \\ \text{subject to} & Ax \geq b(p)/(k+1), \\ & x \geq 0, x \text{ integers }, \end{array}$$

where $b(p) = (N_1(p), \ldots, N_m(p))$ is the (random) vector whose entries are the number of points in each of the squares $p + p_i + [0, \epsilon r)^2$ for $p_i \in [0, Kr)^2 \cap \epsilon r\mathbb{Z}^2$. Let us now consider the LP-relaxation of this program (we drop the condition that the variables need to be integers) and let us denote by $M(p)$ the optimum value of this LP. We remark that as A depends only on ϵ and K, there is a constant $c = c(K, \epsilon)$ such that $\chi(H'_p) \leq M(p) + c(K, \epsilon)$, because rounding up all the variables of a feasible point of the LP-relaxation gives a feasible point of the ILP. By *LP*-duality $M(p)$ is also equal to the value of the program

$$\begin{array}{ll} \max & \frac{1}{k+1} b^T y \\ & \\ \text{subject to} & A^T y \leq 1, \\ & y \geq 0. \end{array}$$

This formulation has the advantage that "the randomness has been moved into the objective function". The polytope defined by $A^T y \leq 1$ depends only on ϵ, K and its vertices $y_1, \ldots, y_m$ are in principle known. We can write $L(p) = \frac{1}{k+1} \max_i y_i^T b(p)$. Now note that the y_i correspond to functions in a natural way. Let $\varphi_i : \mathbb{R}^2 \to [0, 1]$ be the function which is 0 outside $[0, K)^2$ and has value $(y_i)_j$ on the square $x + [0, \epsilon)^2$ if the j-th coordinate of y_i corresponds to the square $p + rx + [0, \epsilon r)^2$. Then

$$y_i^T b(p) = \sum_{j=1}^{n} \varphi_i \left(\frac{X_j - p}{r} \right).$$

Thus for all p we have

$$\chi(H'_p) \leq \frac{1}{k+1} \max_{i=1,\ldots,m} M_{\varphi_i} + c(K, \epsilon).$$

This shows that not only H'_p can be coloured with this many colours for any $p \in \epsilon r\mathbb{Z}^2$, but also the subgraph of G_n induced by the points in the set $W_p := p + [0, Kr)^2 + (K+1)r\mathbb{Z}^2$ (depicted in figure 5). To see this note that if $x \in p + [0, Kr)^2 + (K+1)rz, y \in p + [0, Kr)^2 + (K+1)rz'$ for $z \neq z' \in \mathbb{Z}^2$ then $\| x - y \| \geq r$.

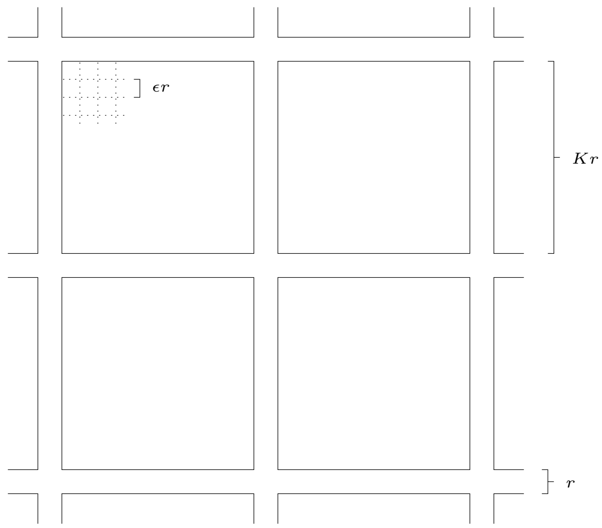

Figure 2: Artists impression of the sets W_p.

We may assume without loss of generality that $K \in \mathbb{N}$. Now consider the collection of all W_p with $p \in r\{0, \ldots, K\}^2$. Note each small square $q + [0, \epsilon r)^2$, $(q \in \epsilon r\mathbb{Z}^2)$, is covered by exactly K^2 of the $(K+1)^2$ sets W_p considered.

Let us now consider the graphs $G'_{n,p}$ which are obtained by replacing each point q of $\Gamma \cap W_p$ by a clique of size $\left\lceil \frac{N(q)}{K^2} \right\rceil$ rather than $N(q)$. The subgraph $G'_{n,p}$ can be k-improperly coloured with no more than

$$\max_p \frac{1}{k+1} \max_i \left(\left\lceil \frac{N_1(p)}{K^2} \right\rceil, \ldots, \left\lceil \frac{N_m(p)}{K^2} \right\rceil \right) \cdot y_i + c(K, \epsilon) \leq \frac{1}{(k+1)K^2} \max_i M_{\phi_i} + c(K, \epsilon) + m$$

colours. The colourings of the $G'_{n,p}$ can be combined to give a colouring of G_n with a total of

$$\frac{1}{k+1} \left(\frac{K+1}{K} \right)^2 \max_{i=1,\ldots,m} M_{\phi_i} + (K+1)^2 (c(K, \epsilon) + m)$$

colours.

Next we wish to lower bound $\chi(G_n)$. For $x \in \mathbb{R}^2$ consider the subgraph H_x of G_n induced by the points in the square $x + [0, rK\frac{1-\epsilon\sqrt{2}}{1+\epsilon\sqrt{2}})^2$. Let us further dissect this square into subsquares of side $\epsilon r\frac{1-\epsilon\sqrt{2}}{1+\epsilon\sqrt{2}}$ in the obvious way and let Γ'_K be the graph with vertices the lower lefthand corners of these subsquares and an edge $p \sim q$ if $\| p - q \| < r(1-\epsilon\sqrt{2})$. Note that Γ'_K is in fact isomorphic to Γ_K and in particular has the same vertex-independent set incidence matrix A. Let H'_x be the graph we get by replacing a vertex of Γ'_K by the number of points in the corresponding subsquare of side $\epsilon r\frac{1-\epsilon\sqrt{2}}{1+\epsilon\sqrt{2}}$. Note H'_x is a subgraph of G_n and $\chi(H'_x)$ is at least the objective value of the linear program

$$\begin{array}{ll} \max & b'(x)^T y \\ \\ \text{subject to} & A^T y \leq 1, \\ & y \geq 0, \end{array}$$

where $b'(x)$ is the vector with the number of points in each of the side $\epsilon r\frac{1-\epsilon\sqrt{2}}{1+\epsilon\sqrt{2}}$ squares. The vertices $y_1, \dots, y_m$ of the polytope are still the same. However, they now correspond to the sums

$$\sum_{j=1}^{n} \varphi'_i\left(\frac{X_j - x}{r}\right),$$

where we set $\varphi'_i(x) := \varphi_i\left(\frac{1+\epsilon\sqrt{2}}{1-\epsilon\sqrt{2}}x\right)$. Maximising over all choices of $x \in \mathbb{R}^2$ we get

$$\chi(G_n) \geq \max_{i=1,\dots,m} M_{\varphi'_i}.$$

To finish the proof note that by Proposition 6 we have that almost surely $M_{\varphi_i}, M_{\varphi'_i} \to \infty$ and $\limsup \frac{M_{\varphi_i}}{M_{\varphi'_i}} \leq \left(\frac{1+\epsilon\sqrt{2}}{1-\epsilon\sqrt{2}}\right)^2$. It follows that

$$1 \leq \limsup \frac{(k+1)\chi^k(G_n)}{\chi(G_n)} \leq \left(\frac{K+1}{K}\right)^2 \left(\frac{1+\epsilon\sqrt{2}}{1-\epsilon\sqrt{2}}\right)^2,$$

and letting $\epsilon, K \to \infty$ now gives the result. □

6 Conclusion

In Sections 2 and 3, we studied the asymptotic behaviour of χ^k when $r \to \infty$ and V is countably infinite. For these results, the bound in Theorem 6 suffices; however, we would be interested to know an exact expression for $\chi^k(G(T, r))$ for any k and r. In Sections 4 and 5, we studied the k-improper chromatic number of random unit disk graphs. A very important issue that we have not yet studied in this respect is the rates of convergence of our results.

In both cases, we have seen that χ^k is well-approximated by the lower bound of Proposition 2. This behaviour differs from that of dense Erdős-Rényi $G(n, p)$ random graphs, where χ^k/χ approaches 1 almost surely. Due to the motivating application in satellite communications, we have focused upon the case with Euclidean norm and dimension two (i.e. unit disk graphs); however, we note here that our results naturally generalise to arbitrary norm and higher dimensions.

A major purpose of this study was to gain insight into the problem of finding the k-improper chromatic number of unit disk graphs. As mentioned in the introduction, the best polynomial approximation for χ^k is 6 (and 3 for $k = 0$). The results of this paper imply for fixed k that, given randomly generated instances G_n, the polynomial computable value $\omega(G_n)/(k+1)$ multiplied by the factor $2\sqrt{3}/\pi$ (which is quite smaller than 6) is a reasonable approximation for the k-improper chromatic number when n is large enough.

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